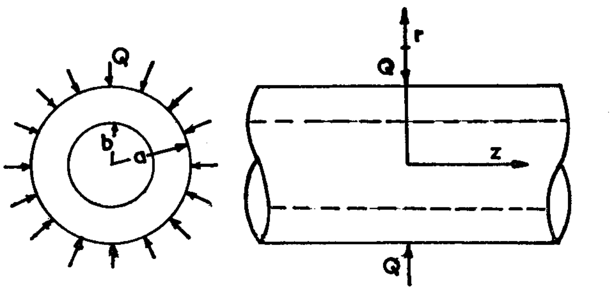

Fig. 1 Long circular-cylindrical shell

suming it as compressive. When the external load is of intensity p over a length 2ξ (in the z direction), $f(z)$ can be expressed as a Fourier-integral, i.e.,

$$f(z) = \frac{2p}{\pi} \int_0^\infty \frac{\sin\alpha\xi}{\alpha} \cos\alpha z d\alpha \tag{5}$$

For a concentrated radial line load Q, one has

$$f(z) = \frac{Q}{\pi} \int_0^\infty \cos\alpha z d\alpha \tag{6}$$

Following the approach for a solid cylinder,[3] the expression for ϕ can be written as follows:

$$\phi = \int_0^\infty \frac{1}{\alpha^3} [A(\alpha)I_0(\alpha r) + B(\alpha)\alpha r I_1(\alpha r) + C(\alpha)K_0(\alpha r) + D(\alpha)\alpha r K_1(\alpha r)] \sin\alpha z d\alpha \tag{7}$$

The stresses are

$$\sigma_r = \int_0^\infty \left\{[(2\nu - 1)B(\alpha) - A(\alpha)]I_0(\alpha r) + \left[\frac{A(\alpha)}{\alpha r} - B(\alpha)\alpha r\right] I_1(\alpha r) + [(1 - 2\nu)D(\alpha) - C(\alpha)]K_0(\alpha r) - \left[\frac{C(\alpha)}{\alpha r} + D(\alpha)\alpha r\right] K_1(\alpha r)\right\}\cos\alpha z d\alpha$$

$$\tau_{rz} = \int_0^\infty \{[A(\alpha) + 2(1 - \nu)B(\alpha)]I_1(\alpha r) + B(\alpha)\alpha r I_0(\alpha r) - [C(\alpha) - 2(1 - \nu)D(\alpha)]K_1(\alpha r) - D(\alpha)\alpha r K_0(\alpha r)\} \sin\alpha z d\alpha \tag{8}$$

$$\sigma_z = \int_0^\infty \{[A(\alpha) + 2(2 - \nu)B(\alpha)]I_0(\alpha r) + B(\alpha)\alpha r I_1(\alpha r) + [C(\alpha) - 2(2 - \nu)D(\alpha)]K_0(\alpha r) + D(\alpha)\alpha r K_1(\alpha r)\} \cos\alpha z\, d\alpha$$

$$\sigma_\theta = \int_0^\infty \left[-A(\alpha) \frac{I_1(\alpha r)}{\alpha r} - (1 - \nu)B(\alpha)I_0(\alpha r) + C(\alpha) \frac{K_1(\alpha r)}{\alpha r} + (1 - \nu)D(\alpha)K_0(\alpha r)\right]\cos\alpha z d\alpha$$

and the displacements are

$$u = \frac{(1 + \nu)}{E} \int_0^\infty \frac{1}{\alpha} \{A(\alpha)I_0(\alpha r) + B(\alpha)[4(1 - \nu)I_0(\alpha r) + \alpha r I_1(\alpha r)] + C(\alpha)K_0(\alpha r) + D(\alpha)[\alpha r K_1(\alpha r) - 4(1 - \nu)K_0(\alpha r)]\} \sin\alpha z d\alpha$$

$$w = \frac{-(1 + \nu)}{E} \int_0^\infty \frac{1}{\alpha} [A(\alpha)I_1(\alpha r) + B(\alpha)\alpha r I_0(\alpha r) - C(\alpha)K_1(\alpha r) - D(\alpha)\alpha r K_0(\alpha r)] \cos\alpha z d\alpha \tag{9}$$

Substitution of the stresses from Eq. (3) into the boundary conditions given in Eqs. (4) and (6) gives the following simultaneous equations, put in the matrix form, to determine $A(\alpha), B(\alpha), C(\alpha)$, and $D(\alpha)$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} A(\alpha) \\ B(\alpha) \\ C(\alpha) \\ D(\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_{45} \end{bmatrix} \tag{10}$$

where

$$\begin{aligned}
a_{11} &= I_1(\alpha a) \\
a_{21} &= I_1(\alpha b) \\
a_{31} &= \frac{I_1(\alpha b)}{\alpha b} - I_0(\alpha b) \\
a_{41} &= \frac{I_1(\alpha a)}{\alpha a} - I_0(\alpha a) \\
a_{13} &= -K_1(\alpha a) \\
a_{23} &= -K_1(\alpha b) \\
a_{33} &= -K_0(\alpha b) - \frac{K_1(\alpha b)}{\alpha b} \\
a_{43} &= -K_0(\alpha a) - \frac{K_1(\alpha a)}{\alpha a} \\
a_{12} &= 2(1 - \nu)I_1(\alpha a) + \alpha a I_0(\alpha a) \\
a_{22} &= 2(1 - \nu)I_1(\alpha b) + \alpha b I_0(\alpha b) \\
a_{32} &= (2\nu - 1)I_0(\alpha b) - \alpha b I_1(\alpha b) \\
a_{42} &= (2\nu - 1)I_0(\alpha a) - \alpha a I_1(\alpha a) \\
a_{14} &= 2(1 - \nu)K_1(\alpha a) - \alpha a K_0(\alpha a) \\
a_{24} &= 2(1 - \nu)K_1(\alpha b) - \alpha b K_0(\alpha b) \\
a_{34} &= (1 - 2\nu)K_0(\alpha b) - \alpha b K_1(\alpha b) \\
a_{44} &= (1 - 2\nu)K_0(\alpha a) - \alpha a K_1(\alpha a) \\
a_{45} &= -Q/\pi
\end{aligned} \tag{11}$$

It can be seen easily that the expressions for stresses and displacements using Eqs. (3) and (10) will be the same as given by Eq. (21) in Ref. 1. Numerical integration to evaluate these stresses and displacements can be done as explained in Ref. 1.

References

[1] Klosner, J. M., "The elasticity solution of a long circular-cylindrical shell subjected to a uniform, circumferential, radial line load," J. Aerospace Sci. **29**, 834–841 (1962).

[2] Tranter, C. J. and Craggs, J. W., "The stress distribution in a long circular cylinder when a discontinuous pressure is applied to the curved surface," Phil. Mag. **36**, 241–250 (1945).

[3] Flügge, W. (ed.), *Handbook of Engineering Mechanics* (McGraw-Hill Book Co., Inc., New York, 1962), pp. 41-15–41-16.

Axisymmetric, Transverse Vibrations of a Spinning Membrane Clamped at Its Center

JAMES G. SIMMONDS*
Massachusetts Institute of Technology, Cambridge, Mass.

By a simple change of scale depending only on Poisson's ratio and membrane geometry, it is shown that the axisymmetric transverse modes of vibration of a fully clamped membrane are equivalent to those of a partially clamped membrane.

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* Graduate Student, Department of Mathematics. Associate Member AIAA.

1. Introduction

In a recent paper,[1] the writer calculated the transverse modes of vibration of a spinning, elastic membrane under the assumption that the membrane was held between two hubs in such a manner that radial but not transverse displacements could occur at the hubs (partial clamping). The static stress field was taken to be that of a freely spinning solid disk, which permitted the equation of motion to be reduced to a hypergeometric equation. In Ref. 1, it was remarked that, if the membrane were fully clamped at the hubs, i.e., if neither radial nor transverse displacements could occur there, the static stress field would be more complicated, and the reduced equation of motion would be Heun's equation—an ordinary second-order differential equation with four regular singular points.

For the axisymmetric modes of a spinning, fully clamped membrane, one of the singularities in Heun's equation disappears, and Legendre's equation is obtained. Below, a simple change of scale is given which reduces this equation and its boundary conditions to a corresponding problem for a partially clamped membrane.

2. Equation of Motion

In the notation of Ref. 1, the differential equation governing axisymmetric vibrations of infinitesimal amplitude is

$$\frac{\partial}{\partial r}\left(r\sigma_r \frac{\partial w}{\partial r}\right) - \rho r \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

For a membrane of radius b fully clamped to a hub of radius a, the radial stress is [see Eq. (53) of Ref. 3, for example]

$$\sigma_r = \frac{3+\nu}{8}\rho\Omega^2(b^2 - r^2)\left(1 + \epsilon\frac{a^2}{r^2}\right) \tag{2}$$

where

$$\epsilon = \frac{1-\nu}{3+\nu}\left[\frac{3+\nu-\lambda^2(1+\nu)}{1+\nu+\lambda^2(1-\nu)}\right] \tag{3}$$

where ν is Poisson's ratio, and $\lambda = a/b$. (For a partially clamped membrane, the radial stress is obtained by setting $\epsilon = 0$.) The boundary conditions associated with Eq. (1) are

$$r = a:\ w = 0 \qquad r = b:\ w = \text{finite} \tag{4}$$

Substituting Eq. (2) into Eq. (1) and setting

$$\frac{w}{b} = y(x)\sin\omega t \qquad x = \frac{1-(r/b)^2}{1+\lambda^2\epsilon} \tag{5}$$

Legendre's differential equation (with singularities at $x = 0$ and $x = 1$) is obtained:

$$\frac{d}{dx}\left[x(1-x)\frac{dy}{dx}\right] + \frac{2\omega^2}{(3+\nu)\Omega^2}y = 0 \tag{6}$$

together with the boundary conditions

$$x = 0:\ y = \text{finite} \qquad x = \frac{1-\lambda^2}{1+\lambda^2\epsilon}:\ y = 0 \tag{7}$$

A comparison of Eqs. (5–7) with those of either Johnson[2] or the writer[1] for the axisymmetric modes of a spinning, partially clamped membrane shows that the two problems are equivalent provided that Λ, the ratio of hub to membrane radii for the partially clamped membrane, be related to λ of the fully clamped membrane by the expression

$$\Lambda^2 = \lambda^2\left(\frac{1+\epsilon}{1+\lambda^2\epsilon}\right) \tag{8}$$

Thus the frequency curves presented by the author[1] and the discussion by Johnson[2] of the case $\Lambda^2 \ll 1$ are directly applicable to the case of a fully clamped membrane.

References

[1] Simmonds, J. G., "The transverse vibrations of a flat spinning membrane," J. Aerospace Sci. **29**, 16–18 (1962).

[2] Johnson, M. W., "On the dynamics of shallow elastic membranes," *Proceedings of the Symposium on the Theory of Thin Elastic Shells*, edited by W. T. Koiter (North-Holland Publishing Co., Amsterdam, 1960), pp. 281–300.

[3] Simmonds, J. G., "The finite deflections of a normally loaded, spinning elastic membrane," J. Aerospace Sci. **29**, 1180–1189 (1962).

Oblique Detonation Waves

Robert A. Gross*
Columbia University, New York, N. Y.

The two-dimensional, steady-flow equations for oblique detonation waves are developed, and solutions for the jump conditions are presented. A proof is presented that the corresponding Chapman-Jouguet condition for oblique detonation waves occurs when the downstream velocity component, normal to the wave, is sonic.

Nomenclature

a = local speed of sound
c_p = specific heat at constant pressure
M = Mach number defined as V/a
p = pressure
Q = heat addition per unit mass of fluid
R = gas constant
T = absolute temperature
V = velocity
γ = ratio of specific heats
δ = flow deflection angle
ρ = density
σ = detonation wave angle

Subscripts

n = component normal to the wave surface
1 = state of the reactant gas upstream of the wave
2 = state of the product gas downstream of the wave

Introduction

The development of high-temperature steady-flow combustion tunnels and the quest for finding methods to permit breathing propulsion systems to obtain very high speeds has generated interest in oblique detonation waves. A detonation wave is an exothermal wave that moves supersonically with respect to the reactant gas. The thickness of the wave and whether it can be treated as a shock wave followed by chemical reactions is a question of wave structure. Only those waves whose thickness is small compared to the extent of the wave surface are considered here, so that the wave may be treated as a discontinuity in the flow.

Experiments on oblique detonation-like waves have been reported by Gross and Chinitz[1] and Rhodes et al.[2] Samaras[3]

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* Professor of Engineering Science, Department of Mechanical Engineering. Member AIAA.